The SIR Model

May 23, 2020

When the blind men had each felt a part of the elephant, the king went to each of them and said to each: "Well, blind man, have you seen the elephant? Tell me, what sort of thing is an elephant?"

The **SIR** epidemiological model divides a total population, S_0 , into three classes, **S**usceptible, **I**nfected and **R**ecovered individuals.[1] They are related by three first-order differential equations,

$$\begin{aligned} \dot{S} &= -k_1 (S/S_0)\, I \\ \dot{I} &= k_1 (S/S_0)\, I - k_2\, I = I\,(k_1 S/S_0 - k_2) \\ \dot{R} &= k_2\, I \end{aligned} \tag{1}$$

involving two rate constants, k_1 , the number of new infections per infector per day and k_2 , the recovery rate of infectors per day. This basic model presumes recovery results either in lifetime immunity or death. These equations may be normalized in terms of the total population, [2]

$$\begin{aligned} x(t) &= S(t)/S_0 \\ y(t) &= I(t)/S_0 \\ z(t) &= R(t)/S_0 \\ x(t) + y(t) + z(t) &= 1 \end{aligned} \tag{2}$$

Thus

$$\frac{dx}{dt} = -k_1 x\, y \quad ; \quad \frac{dy}{dt} = k_1 y\,(x - (k_2/k_1)) \quad ; \quad \frac{dz}{dt} = k_2 y \tag{3}$$

and

$$\frac{dy}{dx} = -1 + \frac{(k_2/k_1)}{x} \tag{4}$$

Given initial conditions $y=0$ and $x=1$,

$$y(x) = 1 - x + (k_2/k_1)\ln(x) \tag{5}$$

At $t=0$, $(\dot{y}/y)_{t=0} = k_1 - k_2$.

At the infection maximum , $x_{max} = k_2/k_1$ and $y_{max} = 1 - (k_2/k_1)(1 - \ln(k_2/k_1))$.

At long times $y(t)$ vanishes as x approaches a limiting value given by $(k_2/k_1)\ln(x_\infty) = x_\infty - 1$.

From values for $y(x)$, the corresponding times may be found by numerical integration,

$$t(x) = -\frac{1}{k_1}\int_{1-\delta}^{x} \frac{d\ln(x')}{y(x')} \tag{6}$$

1 https://en.wikipedia.org/wiki/Compartmental_models_in_epidemiology

2 Solutions may be viewed as trajectories in a plane in 3-dimensional space.

Assuming a value for x_∞, we may calculate k_2/k_1 and thence maximum x, y, and z values *(Fig: 1)*.

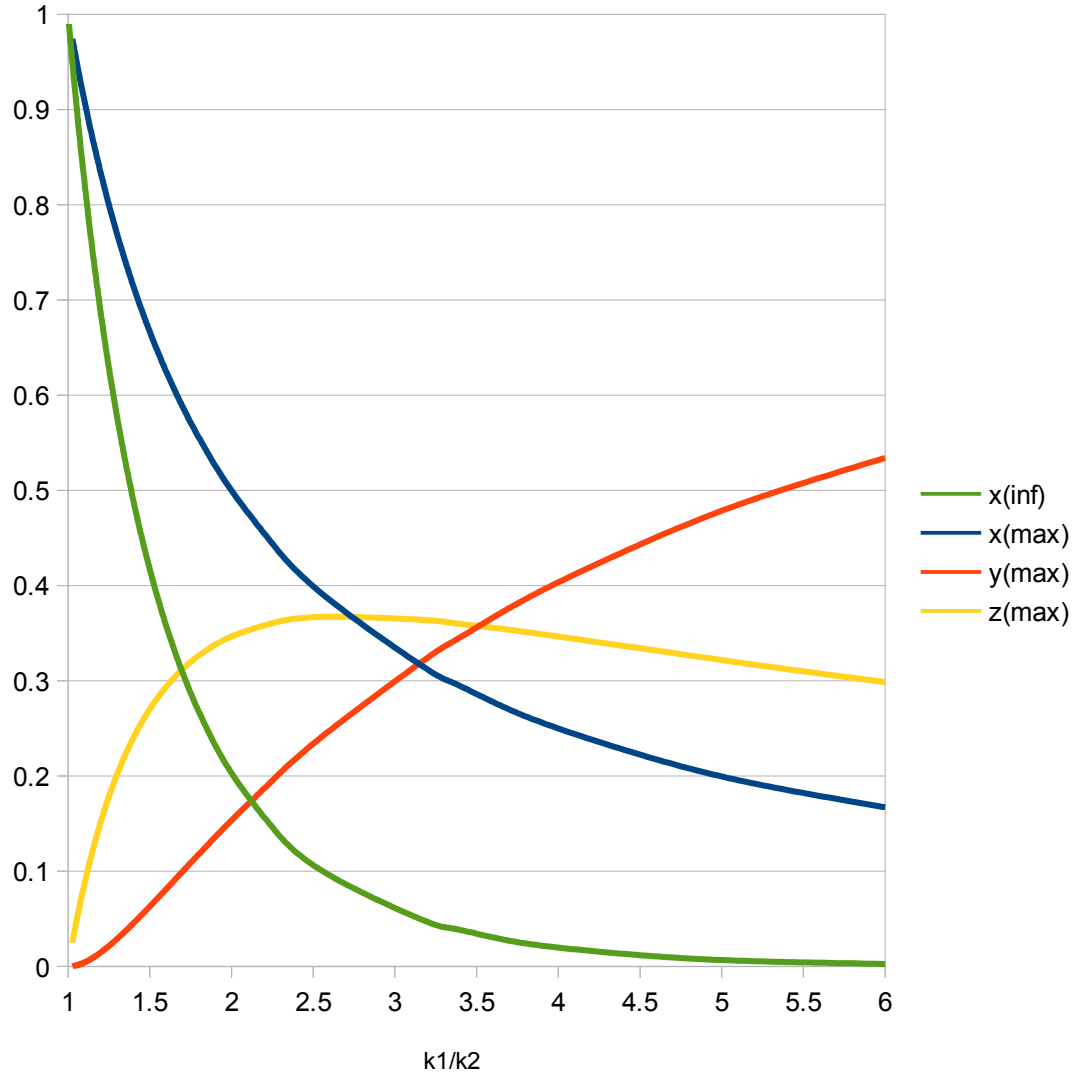

Fig. 1: Variations of SIR functions at the infection maximum plotted versus the rate constant ratio of k_1 and k_2.

Epidemiological behavior requires $k_1/k_2 > 1$, a value equivalent to the number of new infections attributed to one contagious person.[3] Unless this ratio is very close to *1:1*, the SIR model predicts small values for x_∞ , the fraction of the initial population surviving the epidemic. Thus for $k_1/k_2 = 2$, only *20%* of the herd survives infection, for $k_1/k_2 = 4$, only *2%*. In the latter case, the maximum number of active infections is *40%* of the total population with only *25%* not yet infected. No matter the specifics, any epidemic will involve a major fraction of this population.

3 In the literature, this ratio of rate constants is commonly referred to as R_0, the basic reproduction ratio.

To construct an epidemiological plot given k_1 and k_2 , create a geometric progression of decreasing x values, then corresponding y and z values *(Eqs. 5* and *2),* terminating when y turns negative. As $\Delta ln(x)$ is constant, numerical integration of the reciprocal of $y(x)$ to find $t(x)$ is straightforward *(Eq. 6)*.[4] For $k_1=0.4/day$ and $k_2=0.1/day$, *i.e* a recovery time of *10* days while creating *4* infections, we find *98%* of the original population eventually succumbs to infection,[5] and that this final count is 2.4 times the number of infections at the maximum *(Fig. 2*).

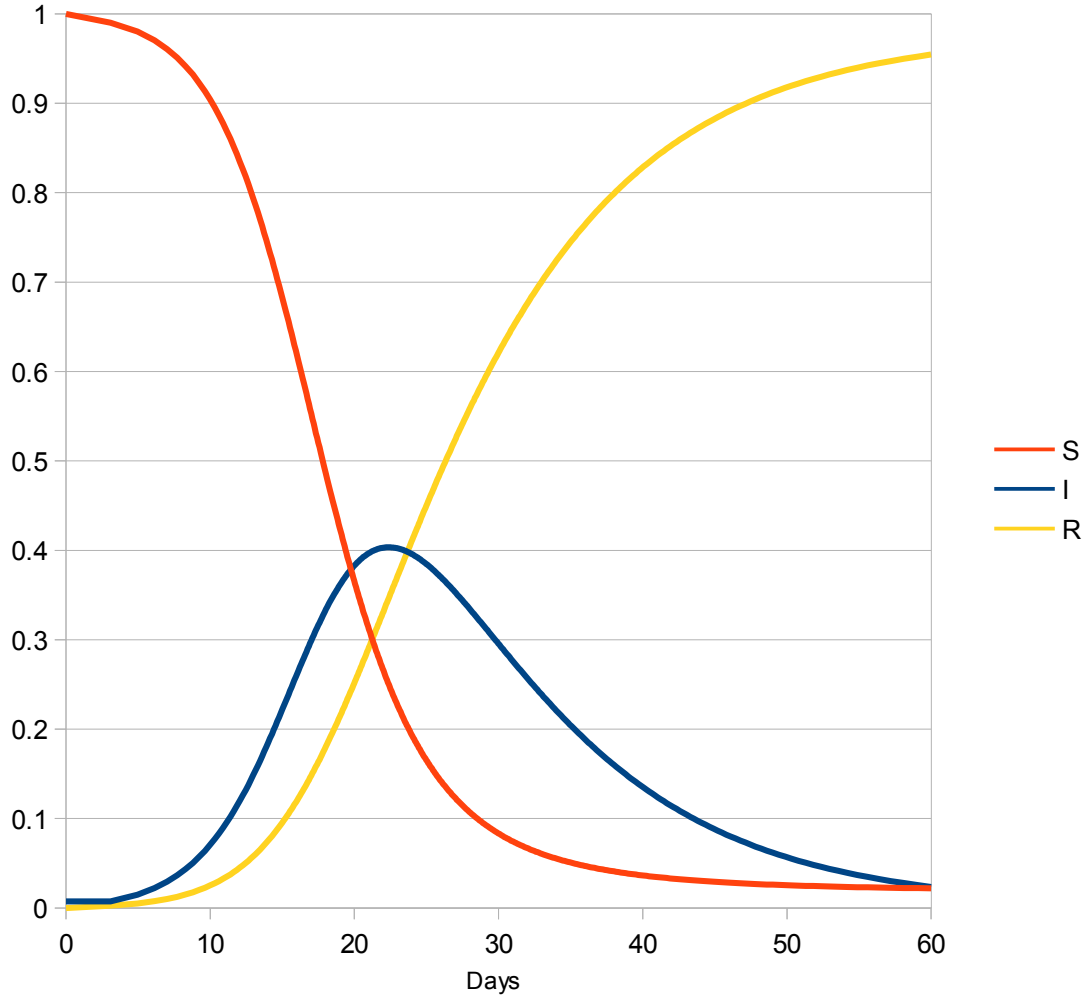

Fig. 2: SIR function plots for $k_1=0.4/day$ and $k_2=0.1/day$.

4 As $y(t)$ vanishes at long and short times, calculated temporal intervals in these regions are less accurate. Indeed without a finite infection at $t=0$, y would remain *0* at all times. Seeding is made by assuming finite decreases in x of *1%* for successive computational increments. Numerical values of x, y and z are exact while the temporal scale in the wings is distorted.

5 These choices are based in part on $\Delta I/I$ values for U.S. data (*vide infra)*. They are also consistent with Chinese COVID-19 estimates of 0.78 and 0.16. The former value depends on a quarantine coefficient and the latter on assumptions of infection duration. *Jianxing He et al., J. Thorac. Dis. 2020;12(3):165-174* (http://dx.doi.org/10.21037/jtd.2020.02.64).

While one parameter, the ratio k_1/k_2, can be deduced from ordinate values of epidemiological curves, the ratio of $\Delta I/I = k_1-k_2$ provides temporal information. COVID values for active cases are reported daily in public sources. *Figure* 3 plots such data, with values ranging *6* to *400,000.* Despite considerable noise, data suggest a common curve with an initial value *ca. 0.3/day* and an abscissa crossing, the long-sought peak, in the weeks of April 12-26.

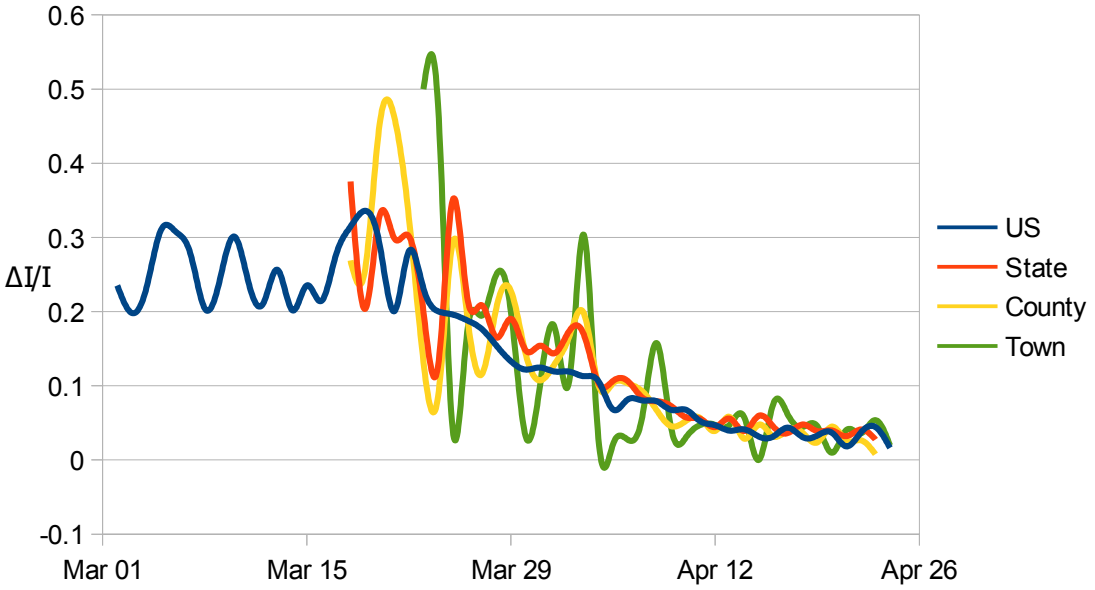

Fig. 3: Empirical data for variations of daily infection changes to total infections.

SIR model theory yields

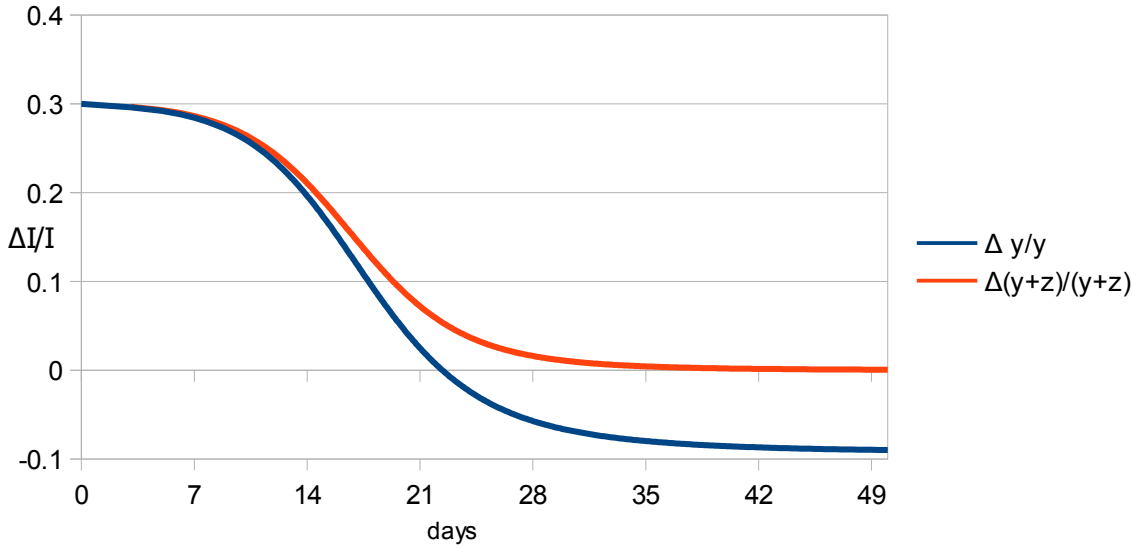

Fig. 4: Theoretical SIR variation of daily infection changes to all active and all infections, past and present, given $k_1=0.4/day$ and $k_2=0.1/day$,

While similar in scales of magnitude and scope, the observed transitions appear broader than this model calculation, possibly reflecting a distribution of starting times. Alternatively, reported values are not of active infections, but total infections, $y+z$. In this case, the function, remains positive with a long-time zero asymptotic value. The short-time asymptote remains k_1-k_2 .

The efficacy of mitigation strategies depends on the k_1/k_2 ratio. Were the initial ratio *5:1*, a two-fold reduction would improve herd survival rate, x_∞, from *0.7%* to *11%*. A further two-fold reduction raises it to 63% , while yet another would take it out of an epidemiological range. An alternative strategy would increase k_2 by quarantine of active cases to shorten the interval over which infections occur. A sensible mitigation policy would quarantine an infected minority.

The usefulness of the SIR model lies in its simplicity. Only two constants are involved – one, a ratio determining the shape of the epidemiological curve, the other its time scale. An obvious modification is to divide the model's three groups into subgroups based on age, sex, race, occupation, *etc*., leading to matrices in place of constants. Another would introduce a coupling between the **S**'s and the **R**'s – the former might directly acquire immunity through vaccination, the latter might lose immunity over time. Another, still, would be to suppose these 'constants' are themselves time-dependent perhaps reflecting seasonal effects. If, for example, $x - k_2/k_1$ should turn positive, an 'echo' epidemic would appear. A successful vaccination campaign will require at least *75%* participation should $k_1/k_2=4$. The SIR model takes no note of spatial inhomogeneities. Should an epidemic have a well-defined starting location, one might expect growth at a rate depending on population density gradients and perimetric expansion.

"With four parameters I can fit an elephant, and with five I can make him wiggle his trunk"
J. v. Neumann (attr.)

Case I

The current COVID-19 pandemic has raised concerns of the consequences of the **R**ecovered group loosing immunity over time to become **S**usceptible. The **SIRS** model introduces a third rate constant, k_3 , describing this relaxation. *Equations 2* become

$$\begin{aligned} dx/dt &= -k_1 x\, y + k_3 z \\ dy/dt &= k_1 x\, y - k_2 y \;=\; y(k_1 x - k_2) \\ dz/dt &= k_2 y - k_3 z \end{aligned} \tag{7}$$

At long times, a steady-state will be reached with

$$\begin{aligned} k_1 x_\infty y_\infty &= k_3 z_\infty \\ k_1 x_\infty &= k_2 \\ k_2 y_\infty &= k_3 z_\infty \end{aligned} \tag{8}$$

Thus,

$$y_\infty = \frac{k_3}{k_1}\cdot\frac{k_1 - k_2}{k_2 + k_3} \tag{9}$$

Figure 5 plots curves calculated for k_1=*0.4*, k_2=*0.1* and k_3 ranging *0.005* to *0.05* day^{-1} corresponding to *200* to *20* days for relaxation.[6] The curves are not significantly changed prior to their maxima and small ripples echo as relaxation proceeds towards a steady-state. The 'recycling' flux is $k_2\, y_\infty$, equivalent to a 147 day period for k_3 = *0.01*. Given a *2%* mortality rate per cycle, lifetime expectancy is reduced to *20* years.

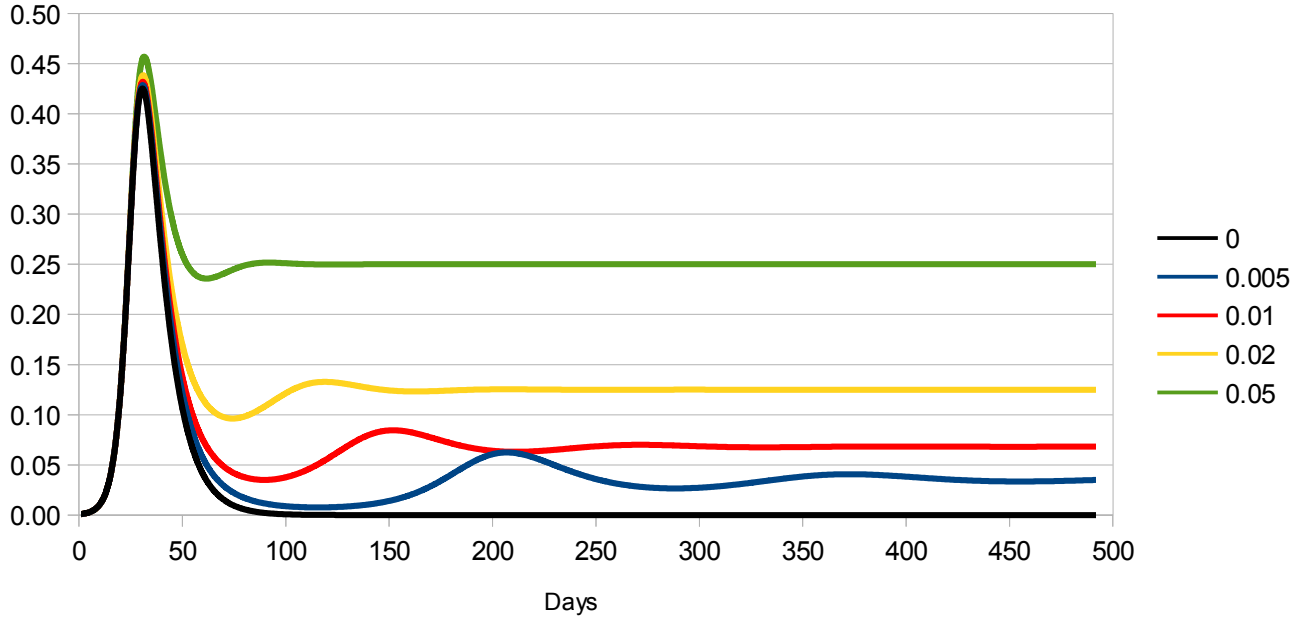

Fig. 5: SIRS plots varying the rate constant, k_3, for loss of immunity.

6 Spread sheet differential summations typically assumed *dt = 1 day* increments.

Case II

The **SEIR** model modifies the basic **SIR** model by introducing an **E**xposed state in which an individual does not reach a contagious phase until an incubation period has passed. The descriptive equations become

$$\begin{aligned} \dot{x} &= -k_1\, x\, y \\ \dot{w} &= k_1\, x\, y - k_3 w \\ \dot{y} &= k_3\, w - k_2\, y \\ \dot{z} &= k_2\, y \end{aligned} \tag{10}$$

where w is the fraction of the total population exposed and carrying the infection but not yet contagious. The constant k_3 now describes the rate at which this transition takes place. *Figure 6* illustrates a typical **SEIR** solution.

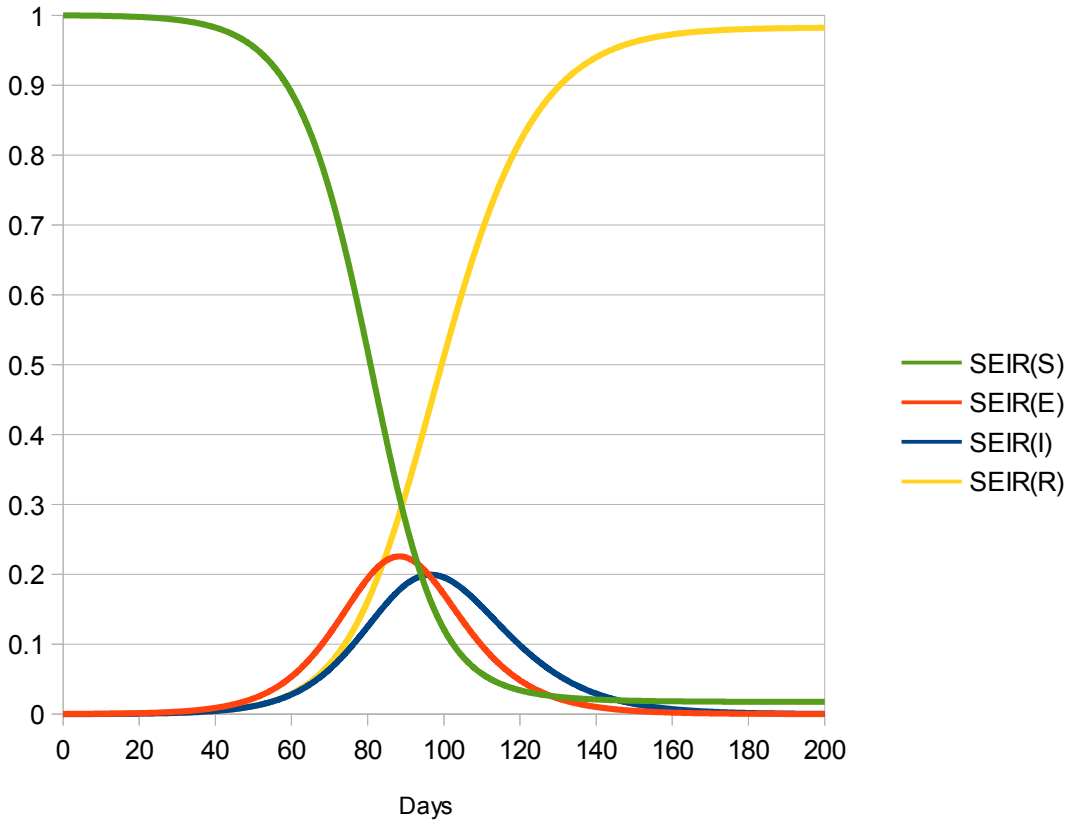

Fig. 6: SEIR plots for k_1=0.4 day^{-1} and k_2=k_3=0.1 day^{-1}.

Analogous to *Eqs. 4* and *5,*

$$\frac{d(w+y)}{dx} = -1 + \frac{(k_2/k_1)}{x} \tag{11}$$

Thus

$$w(x) + y(x) = 1 - x + (k_2/k_1)\ln(x) \tag{12}$$

The asymptotic expression, $(k_2/k_1)\ln(x_\infty) = x_\infty - 1 = -z_\infty$, is identical to that for the **SIR** model, and the long-time recovery parameter, z_∞, is independent of the incubation period.

Short-time asymptotic behavior differs in that the ratio *w/y* approaches a limit given by[7]

$$k_3 u^2 + (k_3 - k_2)u - k_1 = 0 \quad ; \; u = lim\,(w/y)_{t=0} \tag{13}$$

At *t=0*,

$$(\dot{y}/y)_{t=0} = k_3 \frac{w}{y} - k_2 = k_3 u - k_2 \tag{14}$$

For *Fig.6, u=2* and a plot of $\dot{y}/y$ gives

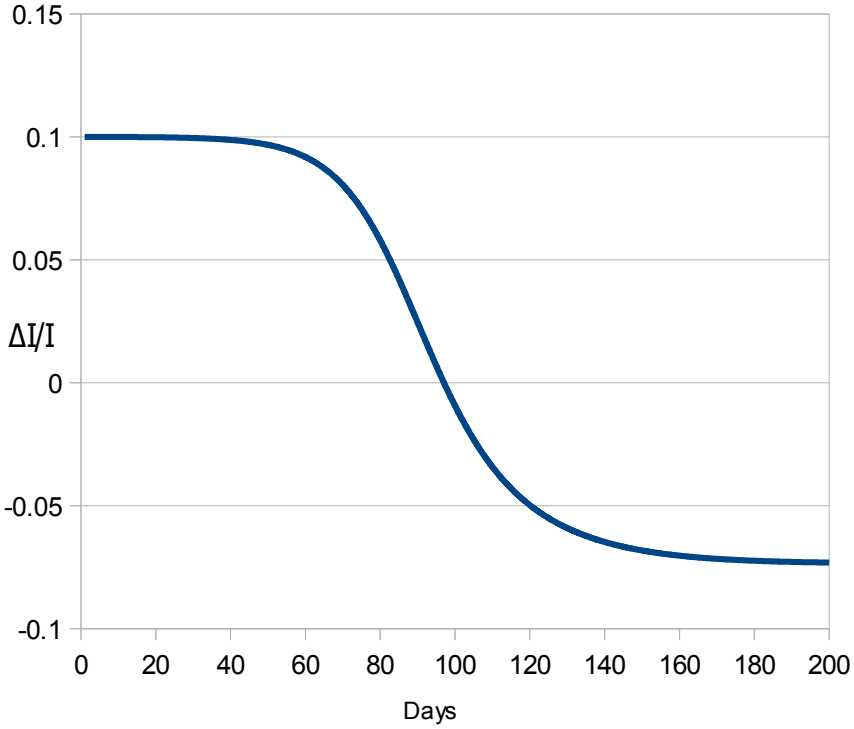

Fig. 7: Variation of daily infection changes to total infections.

Although similar in appearance to an equivalent **SIR** plot *(Fig. 4)*, the *t=0* asymptotic value is no longer simply $k_1 - k_2$.

7 When seeding *y(t=0),* co-seeding *w(t=0)* with *u y(t=0)* will suppress an initial transient spike.

The effects of incubation periods are evident in *Fig. 8* plots of *y(t)*. The longer this period, the longer an epidemic persists. Its amplitude is reduced, although the area beneath these curves remains unchanged.

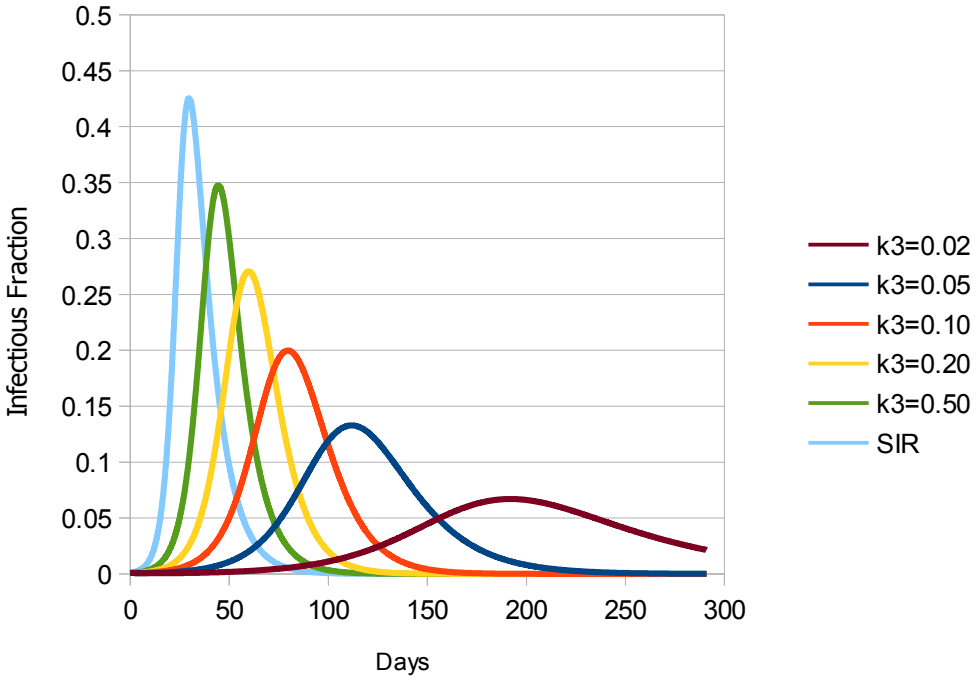

Fig. 8: SEIR curves with k_1=0.4 day^{-1} and k_2=0.1 day^{-1} as functions of k_3 .

At these maxima

$$w_{max} = (k_2/k_3)\, y_{max} \tag{15}$$

and

$$w_{max} + y_{max} = 1 - x_{max} + (k_2/k_1)\ln(x_{max}) \tag{16}$$

so that

$$y_{max} = \frac{1 - x_{max} + (k_2/k_1)\ln(x_{max})}{1 + k_2/k_3} \tag{17}$$

Although the maxima for *w* and *y* do not occur for the same values of *t* and *x,* it is not numerically a bad approximation to assume that $y_{max}(x_{max})$ is also an extremum and thus

$$y_{max} = \frac{1 - (k_2/k_1) + (k_2/k_1)\ln(k_2/k_1)}{1 + k_2/k_3} \tag{18}$$

As $k_3 \to \infty$, the ***SIR*** value is approached.

Case III

The consequences of *Social Distancing* may be modeled by reducing the rate of infection from $k_1=0.4$ for a finite time interval. In *Fig. 9*, this parameter is reduced for *30* days from *Day 20* to *Day 50.* For $k_1=0$, the fraction of active infections drops exponentially from *10%* to *0.04%.* Upon restoring k_1 to its original value, however, infections grow to a new maximum modestly less than what otherwise would have been. For intermediate k_1 values, there is an optimum value for reducing the maximum. The total fraction of the population infected is only slightly altered, ranging *94-98%.*

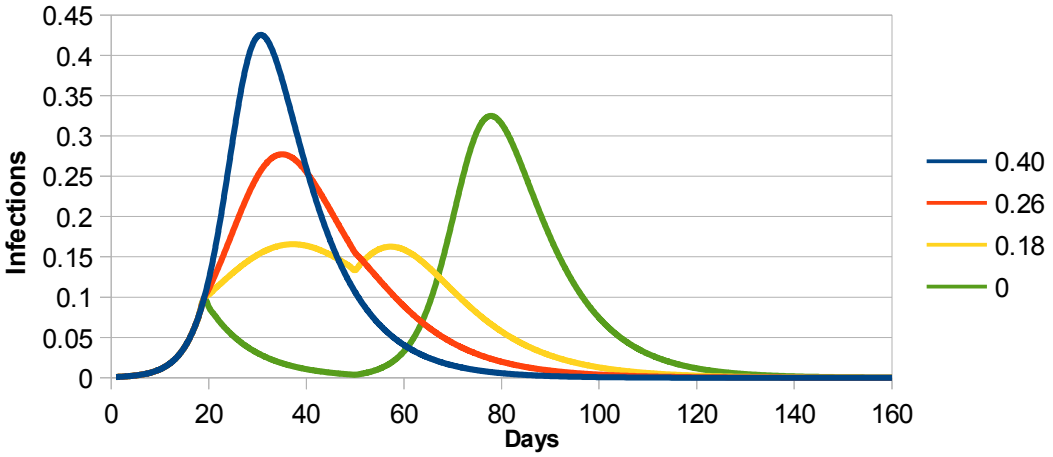

Fig. 9: Emulating Social Distancing by reducing the infection rate constant, k_1 between days 20 and 50.

If, on the other hand, k_1 is reduced or **<u>increased</u>** after passing the peak, there is virtually no significant effect of *Social Distancing* to be seen in the recovery phase *(Fig. 10)*. Even doubling k_1 from its original value only extends the recovery period by a few days.

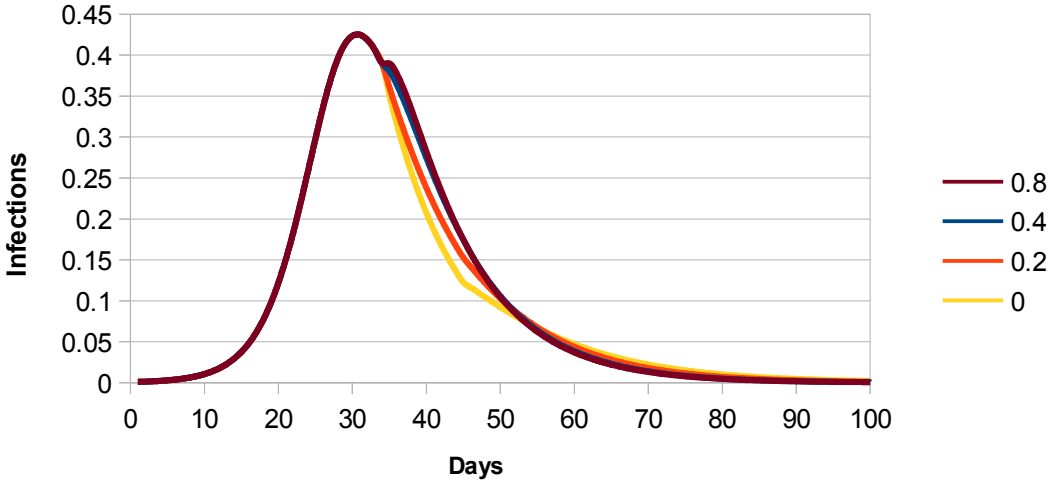

Fig. 10: Emulating Social Distancing by reducing or enhancing the infection rate constant, k_1 between days 35 and 45.